AD-A084 795 FLORIDA STATE UNIV TALLAHASSEE DEPT OF STATISTICS F/G 12/1
A NECESSARY AND SUFFICIENT CONDITION FOR REACHING A CONSENSUS U--ETC(U)
APR 80 R L BERGER DAAG29-79-C-0158
UNCLASSIFIED FSU-STATISTICS-M544 ARO-16713.10-M NL

1 OF 1
AD
A084795

END
DATE
FILMED
6-80
DTIC

A NECESSARY AND SUFFICIENT CONDITION
FOR REACHING A CONSENSUS USING
DeGROOT'S METHOD

Roger L. Berger

FSU Statistics Report M544
USARO Technical Report D-47

The Florida State University
Department of Statistics
Tallahassee, Florida

(12)

A NECESSARY AND SUFFICIENT CONDITION FOR REACHING A CONSENSUS USING DeGROOT'S METHOD

Roger L. Berger

FSU Statistics Report M544
USARO Technical Report D-47

April, 1980
Florida State University
Department of Statistics
Tallahassee, Florida 32306

Authors footnote:

Roger L. Berger is Assistant Professor, Department of Statistics, The Florida State University, Tallahassee, Florida 32306. Research was supported by U.S. Army Research Office Grant DAAG29 79 C 0158.

ABSTRACT

DeGroot (1974) proposed a model in which a group of k individuals might reach a consensus on a common subjective probability distribution for an unknown parameter. This paper presents a necessary and sufficient condition under which a consensus will be reached using DeGroot's method. This work corrects an incorrect statement in the original paper about the conditions needed for a consensus to be reached. The condition for a consensus to be reached is straightforward to check and yields the value of the consensus, if one is reached.

Key words: subjective probability distribution, Markov chain, stochastic matrix, opinion pool.

A Necessary and Sufficient Condition for Reaching a Consensus Using DeGroot's Method

1. INTRODUCTION

Consider a group of k individuals, each of whom can specify his own subjective probability distribution for the unknown value of some parameter θ. Suppose the k individuals must act together as a team or committee. DeGroot (1974) presented a model in which the group might reach a consensus and form a common subjective probability distribution for θ by pooling their opinions. DeGroot's method is both simple and intuitively appealing. For this reason, it has been cited by many authors including Aumann (1976), Dickey and Freeman (1975), Dickey and Gunel (1978), Hogarth (1975), Moskowitz, Schaefer and Borcherding (1976), Ng(1977), Press (1978) and Woodworth (1976).

In this paper, a necessary and sufficient condition is presented under which a consensus will be reached using DeGroot's method. DeGroot presented one such condition but that condition turns out to be sufficient but not necessary. So this paper presents a weaker condition under which a consensus will be reached. The condition which must be checked to determine if a consensus can be reached is explicitly calculated. Roughly speaking, the result is that the group of k individuals can be partitioned into subgroups. The behavior of each subgroup determines whether or not the whole group will reach a consensus.

2. MODEL FOR REACHING A CONSENSUS

DeGroot (1974) presented the following model under which a consensus might be reached among the k individuals. A more detailed explanation of the model can be found in DeGroot's paper.

For $i = 1, \ldots, k$, let F_i denote t e subjective probability distribution which individual i assigns to the parameter θ. The subjective distributions, $F_1, \ldots, F_k$, will be based on the different backgrounds and different levels of expertise of the members of the group. It is assumed that, if individual i is informed of the distributions of each of the other members of the group, he might wish to revise his subjective distribution to accomodate this information. It is further assumed that when individual i makes this revision, his revised distribution is a linear combination of the distributions $F_1, \ldots, F_k$. Let p_{ij} denote the weight that individual i assigns to F_j when he makes this revision. It is assumed that the p_{ij}'s are all nonnegative and $\sum_{j=1}^{k} p_{ij} = 1$. So, after being informed of the subjective distributions of the other members of the group, individual i revises his own subjective distribution from F_i to $F_{i1} = \sum_{j=1}^{k} p_{ij}F_j$.

Let $\underset{\sim}{P}$ denote the $k \times k$ matrix whose (i, j)th element is $p_{ij}(i = 1, \ldots, k;\ j = 1, \ldots, k)$. $\underset{\sim}{P}$ is a stochastic matrix since the elements are all nonnegative and the rows sum to one. Let $\underset{\sim}{F}$ and $\underset{\sim}{F}^{(1)}$ be the vectors whose transposes are $\underset{\sim}{F}' = (F_1, \ldots, F_k)$ and $\underset{\sim}{F}^{(1)'} = (F_{11}, \ldots, F_{k1})$. Then the vector of revised subjective distributions can be written as $\underset{\sim}{F}^{(1)} = \underset{\sim}{P}\underset{\sim}{F}$.

The critical step in this process is that now the above revision is iterated. After being informed of the revised subjective distributions, $F_{11}, \ldots, F_{k1}$, of the other members of the group, it is assumed that individual i now revises his subjective distribution from F_{i1} to $F_{i2} = \sum_{j=1}^{k} p_{ij}F_{j1}$. The process continues in this way. Let F_{in} denote the subjective distribution of individual i after n revisions. Let $\underset{\sim}{F}^{(n)}$ denote the vector whose transpose is $F^{(n)'} = (F_{1n}, \ldots, F_{kn})$. Then $\underset{\sim}{F}^{(n)} = \underset{\sim}{P}\underset{\sim}{F}^{(n-1)} = \underset{\sim}{P}^n\underset{\sim}{F}$, n = 2, 3, ... It is assumed that these revisions are made indefinitely or until $\underset{\sim}{F}^{(n+1)} = \underset{\sim}{F}^{(n)}$ for some n.

DeGroot defines that a consensus is reached if and only if all k components of $\underset{\sim}{F}^{(n)}$ converge to the same limit as $n \to \infty$. That is to say, a consensus is reached if and only if there exists a distribution F* such that $\lim_{n\to\infty} F_{in} = F^*$, $i = 1, \ldots, k$.

DeGroot goes on to assert that a consensus is reached if and only if every row of the matrix $\underset{\sim}{P}^n$ converges to the same vector, say $\underset{\sim}{\pi} = (\pi_1, \ldots, \pi_k)$. This is clearly a sufficient condition for a consensus to be reached. But it is not a necessary condition as can be seen from this simple example. Suppose $F_1 = F_2 = \ldots = F_k$. Then it makes no difference what $\underset{\sim}{P}$ is since $\underset{\sim}{F}^{(n)} = \underset{\sim}{P}^n\underset{\sim}{F} = \underset{\sim}{F}$, n = 2, 3, ... Thus the consensus F_1 is reached no matter what weights p_{ij} are used.

Whether or not a consensus is reached depends not only on $\underset{\sim}{P}$ (as suggested by DeGroot's condition) but also on $\underset{\sim}{F}$. The remainder of this paper explains how to check if a consensus is reached and how to calculate the consensus if one is reached for an arbitrary set of weights $\underset{\sim}{P}$ and an arbitrary set of initial subjective distributions $\underset{\sim}{F}$.

Chatterjee and Seneta (1977) consider a generalization of DeGroot's model in which the individuals can change their weights p_{ij} at each iteration. They consider conditions under which a consensus will be reached using this more general model. But they only consider the situation in which all the rows of the weight matrix converge to a common vector. So they do not take into account the effect of $\underset{\sim}{F}$ on whether or not a consensus is reached.

3. CONDITION FOR CONVERGENCE

Since the matrix $\underset{\sim}{P}$ is a $k \times k$ stochastic matrix is can be regarded as the one-step transition probability matrix of a Markov chain with k states and stationary transition probabilities. With this interpretation, standard results about Markov chains can be applied here. These results will be used freely in this discussion. Standard references such as Chung (1960) and Karlin (1969) may be consulted for statements of these results.

By appropriately relabling the individuals in the group, the matrix $\underset{\sim}{P}$ can be put into this form:

$$\underset{\sim}{P} = \begin{pmatrix} \underset{\sim}{P}_1 & \underset{\sim}{0} & \cdots & \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{P}_2 & \cdots & \underset{\sim}{0} & \underset{\sim}{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \underset{\sim}{0} & \underset{\sim}{0} & \cdots & \underset{\sim}{P}_m & \underset{\sim}{0} \\ & & \underset{\sim}{P}_{m+1} & & \end{pmatrix} .$$

Here $\underset{\sim}{P}_i$ is an $m_i \times m_i$ matrix, $i = 1, \ldots, m$. $\underset{\sim}{P}_{m+1}$ is an $m_{m+1} \times k$ matrix. In this Markov chain there are m recurrent classes of communicating states. States 1 through m_1 form the first recurrent class. States $m_1 + 1$ through $m_1 + m_2$ form the second recurrent class and so on. States $(\sum_{i=1}^{m} m_i) + 1$ through k are the transient states. If there are no transient states in the chain, m_{m+1} is taken to be zero and $\underset{\sim}{P}_{m+1}$ is not in the matrix.

Let d_i denote the period of the ith recurrent class. If the class is aperiodic, $d_i = 1$. Then by appropriately relabeling the individuals in the class, $\underset{\sim}{P}_i$ can be written in the form:

$$\underset{\sim}{P}_i = \begin{pmatrix} \underset{\sim}{0} & \underset{\sim}{P}_{i1} & \underset{\sim}{0} & \cdots & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{P}_{i2} & \cdots & \underset{\sim}{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} & \cdots & \underset{\sim}{P}_{id_{i-1}} \\ \underset{\sim}{P}_{id_i} & \underset{\sim}{0} & \underset{\sim}{0} & \cdots & \underset{\sim}{0} \end{pmatrix}$$

Here $\underset{\sim}{P}_{ij}$ is an $m_{ij} \times m_{i(j+1)}$ matrix, $j = 1, \ldots, d_i$. All of the m_{ij} are positive integers, $m_{i1} = m_{i(d_i+1)}$, and $\sum_{j=1}^{d_i} m_{ij} = m_i$. If the class is aperiodic, let $\underset{\sim}{P}_{i1} = \underset{\sim}{P}_i$ and interpret the above notation as $\underset{\sim}{P}_i = \underset{\sim}{P}_{i1}$. Let $M_1 = 0$ and $M_i = \sum_{j=1}^{i-1} m_j$, $i = 2, \ldots, m$. The states $M_i + 1$ through $M_i + m_{i1}$ are called the first moving subclass of the $i\underline{th}$ recurrent class. The states $M_i + m_{i1} + 1$ through $M_i + m_{i1} + m_{i2}$ are called the second moving subclass of the $i\underline{th}$ recurrent class, and so on.

Then all of the recurrent states in the chain (and hence all of the individuals in the group corresponding to these recurrent states) can be partitioned into subgroups according to which moving subclass they belong to. There are $d = \sum_{i=1}^{m} d_i$ subgroups in this partition.

For $i = 1, \ldots, m$ and $j = 1, \ldots, d_i$, let $\underset{\sim}{A}_{ij}$ denote the $m_{ij} \times m_{ij}$ matrix given by $\underset{\sim}{A}_{ij} = \underset{\sim}{P}_{ij}\, \underset{\sim}{P}_{i(j+1)} \cdots \underset{\sim}{P}_{id_i}\, \underset{\sim}{P}_{i1} \cdots \underset{\sim}{P}_{i(j-1)}$.

Then $\underset{\sim}{P}_i^{d_i}$ is given by

$$\underset{\sim}{P}_i^{d_i} = \begin{pmatrix} \underset{\sim}{A}_{i1} & \underset{\sim}{0} & \cdots & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{A}_{i2} & \cdots & \underset{\sim}{0} \\ \vdots & \vdots & \ddots & \vdots \\ \underset{\sim}{0} & \underset{\sim}{0} & \cdots & \underset{\sim}{A}_{id_i} \end{pmatrix}.$$

Let $\underset{\sim}{\pi}(i, j) = (\pi(i, j)_1, \ldots, \pi(i, j)_{m_{ij}})$ be the solution to the linear equations $\underset{\sim}{\pi}(i, j)\underset{\sim}{A}_{ij} = \underset{\sim}{\pi}(i, j)$ together with the equation $\sum_{\ell=1}^{m_{ij}} \pi(i, j)_\ell = 1$. Since $\underset{\sim}{A}_{ij}$ is the one-step transition probability matrix for an irreducible aperiodic Markov chain, a solution $\underset{\sim}{\pi}(i, j)$ exists and it is unique. Let $\underset{\sim}{F}(i, j)$ denote the $m_{ij} \times 1$ vector of initial subjective probability distributions for the individuals in the $j\underline{th}$ moving subclass of the $i\underline{th}$ recurrent class. That is, $\underset{\sim}{F}(i, j)$ is the vector whose transpose is $\underset{\sim}{F}'(i, j) = (F_{M_{ij}+1}, \ldots, F_{M_{ij}+m_{ij}})$

where $M_{ij} = (\sum_{\ell=1}^{i-1} m_\ell) + (\sum_{\ell=1}^{j-1} m_{i\ell})$ and any sum from one to zero is defined to be zero.

Now the necessary and sufficient condition for a consensus to be reached can be stated. Theorem 1 gives the limiting distribution for a recurrent individual if such a limit exists. Theorem 2 gives the necessary and sufficient condition for the group to reach a consensus. The proofs of both theorems are given in Section 6.

Theorem 1: If individual ℓ is in the jth moving subclass of the ith recurrent class and if $\lim_{n\to\infty} F_{\ell n}$ exists then $\lim_{n\to\infty} F_{\ell n} = \underset{\sim}{\pi}(i, j)\underset{\sim}{F}(i, j)$.

Theorem 2: a) If $d = 1$, a consensus is reached and the consensus is $\underset{\sim}{\pi}(1, 1)\underset{\sim}{F}(1, 1)$.

b) If $d > 1$, a consensus is reached if and only if $\underset{\sim}{\pi}(i, j)\underset{\sim}{F}(i, j) = F^*$ for every $i = 1, \ldots, m$; $j = 1, \ldots, d_i$, for some distribution F^*. The consensus, if it is reached, is F^*.

The case a) $d = 1$ is the case considered by DeGroot for, in this situation, all of the rows of $\underset{\sim}{P}^n$ converge to the vector $(\underset{\sim}{\pi}(1, 1)\ \underset{\sim}{0})$ where $\underset{\sim}{0}$ is a $1 \times m_2$ vector of zeros and m_2 is the number of transient states. But case b) $d > 1$ gives the condition under which a consensus will be reached in the situation in which DeGroot claimed that a consensus would not be reached, namely, if there are at least two disjoint classes of communicating states or at least one class of communicating states is periodic.

4. AN EXAMPLE

The notation of Section 3 and the results of Theorems 1 and 2 will be illustrated with the following example. Suppose k = 8 and

$$\underset{\sim}{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} \end{pmatrix}.$$

Then $m = 2$, $d_1 = 1$, $d_2 = 2$, and $d = d_1 + d_2 = 3$.

$$\underset{\sim}{P}_1 = \underset{\sim}{P}_{11} = \underset{\sim}{A}_{11} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

and $\underset{\sim}{\pi}(1, 1)$, the solution to $\underset{\sim}{\pi}(1, 1)\underset{\sim}{A}_{11} = \underset{\sim}{\pi}(1, 1)$ and $\sum_{\ell=1}^{3} \pi(1, 1)_\ell = 1$, is $(\frac{4}{11}, \frac{3}{11}, \frac{4}{11})$.

$$\underset{\sim}{P}_2 = \begin{pmatrix} \underset{\sim}{0} & \underset{\sim}{P}_{21} \\ \underset{\sim}{P}_{22} & \underset{\sim}{0} \end{pmatrix} \qquad \text{where}$$

$$\underset{\sim}{P}_{21} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} \quad \text{and} \quad \underset{\sim}{P}_{22} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} .$$

$$\underset{\sim}{A}_{21} = \begin{pmatrix} \frac{5}{12} & \frac{7}{12} \\ \frac{11}{24} & \frac{13}{24} \end{pmatrix} \quad \text{and} \quad \underset{\sim}{A}_{22} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{3}{8} & \frac{5}{8} \end{pmatrix} .$$

Solving the linear equations yields $\underset{\sim}{\pi}(2, 1) = (\frac{11}{25}, \frac{14}{25})$ and $\underset{\sim}{\pi}(2, 2) = (\frac{9}{25}, \frac{16}{25})$. Theorem 2 states that a consensus is reached if and only if $\frac{4}{11} F_1 + \frac{3}{11} F_2 + \frac{4}{11} F_3 = \frac{11}{25} F_4 + \frac{14}{25} F_5 = \frac{9}{25} F_6 + \frac{16}{25} F_7$. The consensus, if it is reached, is the common value. In this example, the eighth state is transient and has no effect on whether or not a consensus is reached. Also, F_8 does not enter into the calculation of the consensus.

5. A COMPUTATIONAL SHORTCUT

To determine if a consensus is reached, it is necessary to compute the vectors $\underset{\sim}{\pi}(i, j)$ $(i = 1, \ldots, m;\ j = 1, \ldots, d_i)$. Each of these vectors is defined as the solution of a certain set of linear equations. The following result states that, for each $i = 1, \ldots, m$, it is only necessary to solve the linear equations for $\underset{\sim}{\pi}(i, 1)$. The remaining $d_i - 1$ vectors, $\underset{\sim}{\pi}(i, 2), \ldots, \underset{\sim}{\pi}(i, d_i)$, can be determined by simple matrix multiplication.

<u>Theorem 3</u>: For any $i = 1, \ldots, m$ and $j = 2, \ldots, d_i$, $\underset{\sim}{\pi}(i, j) = \underset{\sim}{\pi}(i, j-1)\underset{\sim}{P}_{i(j-1)}$.

<u>Remark</u>: For example, in the previous example it is easily verified that $\underset{\sim}{\pi}(2, 2) = (\frac{9}{25}, \frac{16}{25}) = \underset{\sim}{\pi}(2, 1)\underset{\sim}{P}_{21}$.

<u>*Proof:*</u> *It suffices to show that* $\underset{\sim}{\pi}(i, j-1)\underset{\sim}{P}_{i(j-1)}$ satisfies the appropriate linear equalities, i.e., the sum of the coordinates of $\underset{\sim}{\pi}(i, j-1)\underset{\sim}{P}_{i(j-1)}$ is one and $\underset{\sim}{\pi}(i, j-1)\underset{\sim}{P}_{i(j-1)}\underset{\sim}{A}_{ij} = \underset{\sim}{\pi}(i, j-1)\underset{\sim}{P}_{i(j-1)}$. The sum of the coordinates is one since the sum of the coordinates of $\underset{\sim}{\pi}(i, j-1)$ is one and the sum of each row of $\underset{\sim}{P}_{i(j-1)}$ is one. The definition of $\underset{\sim}{A}_{i(j-1)}$ and $\underset{\sim}{A}_{ij}$ and the fact that $\underset{\sim}{\pi}(i, j-1)\underset{\sim}{A}_{i(j-1)} = \underset{\sim}{\pi}(i, j-1)$ yields

$$\begin{aligned}\underset{\sim}{\pi}(i, j-1)\underset{\sim}{P}_{i(j-1)}\underset{\sim}{A}_{ij} &= \underset{\sim}{\pi}(i, j-1)\underset{\sim}{P}_{i(j-1)}(\underset{\sim}{P}_{ij} \cdots \underset{\sim}{P}_{id_i}\underset{\sim}{P}_{i1} \cdots \underset{\sim}{P}_{i(j-1)}) \\ &= \underset{\sim}{\pi}(i, j-1)\underset{\sim}{A}_{i(j-1)}\underset{\sim}{P}_{i(j-1)} \\ &= \underset{\sim}{\pi}(i, j-1)\underset{\sim}{P}_{i(j-1)} .\end{aligned}$$

Hence the second equality is also true. ||

6. PROOFS OF THEOREMS 1 AND 2

Let $\underset{\sim}{p}_i^{(n)}$ denote the ith row of $\underset{\sim}{P}^n$, $i = 1, \ldots, k$. Let $\underset{\sim}{0}_j$ denote a $1 \times j$ vector of zeros. All of the limiting results for stochastic matrices used in these two proofs are summarized in Part I, Section 6, Theorem 4 of Chung (1960).

Proof of Theorem 1: Suppose ℓ is in the jth moving subclass of the ith recurrent class. Then $\lim_{n\to\infty} \underset{\sim}{p}_\ell^{(nd_i)}$ exists and is equal to $p_\ell^* = (\underset{\sim}{0}_{M_{ij}}\ \underset{\sim}{\pi}(i, j)\ \underset{\sim}{0}_{k-M_{ij}-m_{ij}})$.

So $\lim_{n\to\infty} F_{\ell(nd_i)} = \lim_{n\to\infty} \underset{\sim}{p}_\ell^{(nd_i)}\ \underset{\sim}{F} = \underset{\sim}{p}_\ell^*\ \underset{\sim}{F} = \underset{\sim}{\pi}(i, j)\underset{\sim}{F}(i, j)$. If $\lim_{n\to\infty} F_{\ell n}$ exists, it must equal the limit of the subsequence $F_{\ell(nd_i)}$. Therefore $\lim_{n\to\infty} F_{\ell n} = \underset{\sim}{\pi}(i, j)\underset{\sim}{F}(i, j)$. ||

Proof of Theorem 2: a) If $d = 1$ then there is only one recurrent class and it is aperiodic. So $\lim_{n\to\infty} \underset{\sim}{p}_i^{(n)}$ exists and equals $\underset{\sim}{p}^* = (\underset{\sim}{\pi}(1, 1)\ \underset{\sim}{0}_{m_2})$ for every $i = 1, \ldots, k$. Thus $\lim_{n\to\infty} F_{in} = \lim_{n\to\infty} \underset{\sim}{p}_i^{(n)}\ \underset{\sim}{F} = \underset{\sim}{p}^*\ \underset{\sim}{F} = \underset{\sim}{\pi}(1, 1)\underset{\sim}{F}(1, 1)$ for every $i = 1, \ldots, k$. So a consensus is reached and the consensus is $\underset{\sim}{\pi}(1, 1)\underset{\sim}{F}(1, 1)$.

b) (Necessity) Suppose a consensus is reached. Then $\lim_{n\to\infty} F_{in} = F^*$ for every $i = 1, \ldots, k$. If ℓ is in the jth moving class of the ith recurrent class, by Theorem 1, $\underset{\sim}{\pi}(i, j)\underset{\sim}{F}(i, j) = \lim_{n\to\infty} F_{\ell n} = F^*$. Thus $\underset{\sim}{\pi}(i, j)\underset{\sim}{F}(i, j) = F^*$ $(i = 1, \ldots, m;\ j = 1, \ldots, d_i)$.

b) (Sufficiency) Suppose $\underset{\sim}{\pi}(i, j)\underset{\sim}{F}(i, j) = F^*$ $(i = 1, \ldots, m,\ j = 1, \ldots, d_i)$.

First it will be shown that, if ℓ is a recurrent state, $\lim_{n\to\infty} F_{\ell n}$ exists and equals F^*. Suppose ℓ is in the jth moving subclass of the ith recurrent class. Then, for $r = 0, \ldots, d_i - 1$, $\lim \underset{\sim}{p}_\ell^{(nd_i+r)}$ exists and equals $\underset{\sim}{p}_\ell^*(r) = (\underset{\sim}{0}_{M_{iq}}\ \underset{\sim}{\pi}(i, q)\underset{\sim}{0}_{k-M_{iq}-m_{iq}})$ where $q = (j + r)(\mathrm{mod}\ d_i)$. (Note, here

$M_{i0} = M_{id_i}$, $m_{i0} = m_{id_i}$, $\underset{\sim}{\pi}(i, 0) = \underset{\sim}{\pi}(i, d_i)$ and $\underset{\sim}{F}(i, 0) = \underset{\sim}{F}(i, d_i)$ for $i = 1, \ldots, m$.) Thus $\lim_{n\to\infty} F_{\ell(nd_i+r)} = \lim_{n\to\infty} \underset{\sim}{p}_\ell^{(nd_i+r)} \underset{\sim}{F} = \underset{\sim}{p}_\ell^*(r)\underset{\sim}{F} = \underset{\sim}{\pi}(i, q)\underset{\sim}{F}(i, q) = F^*$. Since each of the d_i subsequences $F_{\ell(nd_i+r)}$, $r = 0, \ldots, d_i - 1$, converges to F^*, the full sequence $F_{\ell n}$ also converges to F^*. Thus, since ℓ was an arbitrary recurrent state, every subjective distribution corresponding to a recurrent state converges to F^*.

Finally, it will be shown that if ℓ is a transient state, $\lim_{n\to\infty} F_{\ell n}$ exists and equals F^*. Let $\delta = \Pi_{i=1}^m d_i$. Then, for $r = 0, \ldots, \delta - 1$, $\lim_{n\to\infty} \underset{\sim}{p}_\ell^{(n\delta+r)}$ exists and equals $\underset{\sim}{p}_\ell^*(r) = (f_{\ell 11}^*(r)\underset{\sim}{\pi}(1, 1), f_{\ell 12}^*(r)\underset{\sim}{\pi}(1, 2), \ldots, f_{\ell m d_m}^*(r)\underset{\sim}{\pi}(m, d_m), \underset{\sim}{0}_{m_{m+1}})$ where $f_{\ell ij}^*(r)$ is the probability that the chain is in the j<u>th</u> moving subclass of the i<u>th</u> recurrent class for some $n \equiv r \pmod{d_i}$ given that the chain started in state ℓ. (Note, the fact that the $f_{ij}^*(r)$, as defined by Chung, are constant for j in a particular moving subclass was used to express $\underset{\sim}{p}_\ell^*(r)$ in terms of the $f_{\ell ij}^*(r)$. Also note that $\sum_{i=1}^m \sum_{j=1}^{d_i} f_{\ell ij}^*(r) = 1$.). Thus,

$$
\begin{aligned}
\lim_{n\to\infty} F_{\ell(n\delta+r)} &= \lim_{n\to\infty} \underset{\sim}{p}_\ell^{(n\delta+r)}\underset{\sim}{F} \\
&= \underset{\sim}{p}_\ell^*(r)\underset{\sim}{F} \\
&= \sum_{i=1}^m \sum_{j=1}^{d_i} f_{\ell ij}^*(r)\underset{\sim}{\pi}(i, j)\underset{\sim}{F}(i, j) \\
&= \sum_{i=1}^m \sum_{j=1}^{d_i} f_{\ell ij}^*(r)F^* \\
&= F^* \sum_{i=1}^m \sum_{j=1}^{d_i} f_{\ell ij}^*(r) \\
&= F^*.
\end{aligned}
$$

Since each of the δ subsequences $F_{\ell(n\delta+r)}$, $r = 0, \ldots, \delta - 1$, converges to F^*, the full sequence $F_{\ell n}$ also converges to F^*. Thus, since ℓ was an arbitrary transient state, every subjective distribution corresponding to a transient state converges to F^*. ||

REFERENCES

Aumann, Robert J. (1976), "Agreeing to Disagree," The Annals of Statistics, 4, 1236-1239.

Chatterjee, S. and Seneta, E. (1977), "Towards Consensus: Some Convergence Theorems on Repeated Averaging," Journal of Applied Probability, 14, 89-97.

Chung, Kai Lai (1960), Markov Chains with Stationary Transition Probabilities, Berlin: Springer-Verlag.

DeGroot, Morris H. (1974), "Reaching a Consensus," Journal of the American Statistical Association, 69, 118-121.

Dickey, James and Freeman, Peter (1975), "Population Distributed Personal Probabilities," Journal of the American Statistical Association, 70, 362-364.

__________ and Gunel, E. (1978)," Bayes Factors from Mixed Probabilities," Journal of the Royal Statistical Society, Series B, 40, 43-46.

Hogarth, Robin M. (1975), "Cognitive Processes and the Assessment of Subjective Probability Distributions," Journal of the American Statistical Association, 70, 271-289.

Karlin, Samuel (1969), A First Course in Stochastic Processes, New York: Academic Press.

Moskowitz, Herbert, Schaefer, Ralf E. and Borcherding, Katrin (1976), "Irrationality of Managerial Judgements: Implications for Information Systems," Omega, The International Journal of Management Science, 4, 125-140.

Ng, David S. (1977), "Pareto Optimality of Authentic Information," The Journal of Finance, 32, 1717-1728.

Press, S. James (1978), "Qualitative Controlled Feedback for Forming Group Judgements and Making Decisions," Journal of the American Statistical Association, 73, 526-535.

Woodworth, George G. (1976), "t for Two, or Preposterior Analysis for Two Decision Makers: Interval Estimates for the Mean," The American Statistician, 30, 168-171.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1. REPORT NUMBER
FSU No. M544
USARO No. D-47

2. GOVT ACCESSION NO.
AD-A084 775

3. RECIPIENT'S CATALOG NUMBER

4. TITLE
6 A Necessary and Sufficient Condition for Reaching a Consensus Using DeGroot's Method.

5. TYPE OF REPORT & PERIOD COVERED
7 Technical Report.

6. PERFORMING ORG. REPORT NUMBER
FSU Statistics Report M544

7. AUTHOR(s)
10 Roger L. Berger

8. CONTRACT OR GRANT NUMBER(s)
15 USARO DAAG29-79-C-0158

9. PERFORMING ORGANIZATION NAME & ADDRESS
The Florida State University
Department of Statistics
Tallahassee, Florida 32306

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
12

11. CONTROLLING OFFICE NAME & ADDRESS
U.S. Army Research Office-Durham
P.O. Box 12211
Research Triangle Park, N.C. 27709

12. REPORT DATE
11 Apr[illegible]980

13. NUMBER OF PAGES
13

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)

15. SECURITY CLASS (of this report)
Unclassified

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this report)
Approved for public release, distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from report)

18. SUPPLEMENTARY NOTES

19 KEY WORDS
subjective probability distribution, Markov chain, stochastic matrix, opinion pool.

20. ABSTRACT

DeGroot (1974) proposed a model in which a group of k individuals might reach a consensus on a common subjective probability distribution an unknown parameter. This paper presents a necessary and sufficient condition under which a consensus will be reached using DeGroot's method. This work corrects an incorrect statement in the original paper about the conditions needed for a consensus to be reached. The condition for a consensus to be reached is straightforward to check and yields the value of the consensus, if one is reached.

400277

JOB